Problem 1: Traveling Waves [30 pts]

Consider two sinusoidal waves given by the following functions:

$$E_1(z,t) = E_0 \cos(10t - 20z + \frac{\pi}{2})$$

$$E_2(z,t) = -E_0 \cos(10t + 20z + \frac{\pi}{3})$$

a **Determine the direction of propagation and phase velocity for each of the two waves.**
Consider a single point on the wave (constant phase front), i.e.

$$\omega t - \beta z + \phi_0 = \text{const.}$$

So

$$z = \frac{1}{\beta}(\omega t + \phi_0 - \text{const.})$$

To get the propagation speed of the phase front, take the first time derivative:

$$\frac{\partial z}{\partial t} = \frac{\omega}{\beta} = u_p$$

Both ω and β of $E_1(z,t)$ are greater than zero, so the velocity is positive and thus the wave is propagating in the $\boxed{\text{+z direction}}$.

$$u_{p1} = \frac{\omega_1}{\beta_1} = \frac{10}{20} = \frac{1}{2} \text{ m/s}$$

For $E_2(z,t)$, ω is positive and β is negative, so the velocity is negative and thus the wave is propagating in the $\boxed{\text{-z direction}}$.

$$u_{p2} = \frac{\omega_2}{\beta_2} = \frac{10}{-20} = -\frac{1}{2} \text{ m/s}$$

b **Show that both waves satisfy the wave equation, given below:**

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{u_p^2}\frac{\partial^2 E}{\partial t^2} = 0 \tag{1}$$

Plug in our expression for $E(z,t)$ and see if the equality holds.
Spatial Derivatives:

$$\frac{\partial E_1}{\partial z} = 20E_0 \sin(10t - 20z + \frac{\pi}{2}) \quad \text{and} \quad \frac{\partial E_2}{\partial z} = 20E_0 \sin(10t + 20z + \frac{\pi}{3})$$

$$\frac{\partial^2 E_1}{\partial z^2} = -400E_0 \cos(10t - 20z + \frac{\pi}{2}) \quad \text{and} \quad \frac{\partial^2 E_2}{\partial z^2} = 400E_0 \cos(10t + 20z + \frac{\pi}{3})$$

Time Derivatives:

$$\frac{\partial E_1}{\partial t} = -10E_0 \sin(10t - 20z + \frac{\pi}{2}) \quad \text{and} \quad \frac{\partial E_2}{\partial t} = 10E_0 \sin(10t + 20z + \frac{\pi}{3})$$

$$\frac{\partial^2 E_1}{\partial t^2} = -100E_0 \cos(10t - 20z + \frac{\pi}{2}) \quad \text{and} \quad \frac{\partial^2 E_2}{\partial t^2} = 100E_0 \cos(10t + 20z + \frac{\pi}{3})$$

Substituting these results into equation 1

$$E_1: \quad -400E_0 \cos(10t - 20z + \frac{\pi}{2}) - \frac{1}{u_{p1}^2}\left(-100E_0 \cos(10t - 20z + \frac{\pi}{2})\right) = 0$$

$$E_2: \quad 400E_0 \cos(10t + 20z + \frac{\pi}{3}) - \frac{1}{u_{p2}^2}\left(100E_0 \cos(10t + 20z + \frac{\pi}{3})\right) = 0$$

We know from part a that $u_{p1} = \frac{1}{2}$ and $u_{p2} = -\frac{1}{2}$, making this substitution

$$E_1: \quad -400E_0 \cos(10t - 20z + \frac{\pi}{2}) - 4\left(-100E_0 \cos(10t - 20z + \frac{\pi}{2})\right) = 0$$

$$E_2: \quad 400E_0 \cos(10t + 20z + \frac{\pi}{3}) - 4\left(100E_0 \cos(10t + 20z + \frac{\pi}{3})\right) = 0$$

Thus, both $E_1(z,t)$ and $E_2(z,t)$ satisfy the wave equation!

c **Find the z locations where the waves interfere constructively and destructively (where their superposition is maximal and minimal) at $t = \pi/20$ s.**

First, we find the function for each wave at the specified instantaneous time:

$$E_1(z, t = \frac{\pi}{20}) = E_0 \cos(10(\frac{\pi}{20}) - 20z + \frac{\pi}{2}) = E_0 \cos(-20z + \pi)$$

$$E_2(z, t = \frac{\pi}{20}) = -E_0 \cos(10(\frac{\pi}{20}) + 20z + \frac{\pi}{3}) = -E_0 \cos(20z + \frac{5\pi}{6})$$

We can find constructive and destructive interference when $E_1 - E_2 = 0$:

$$E_0 \cos(-20z + \pi) + E_0 \cos(20z + \frac{5\pi}{6}) = 0$$

$$\cos(-20z + \pi) = -\cos(20z + \frac{5\pi}{6})$$

$$\cos(-20z + \pi) = \cos(20z + \frac{11\pi}{6})$$

Our solution for constructive interference will have the form:

$$-20z + \pi = 2\pi n + 20z + \frac{11\pi}{6}$$

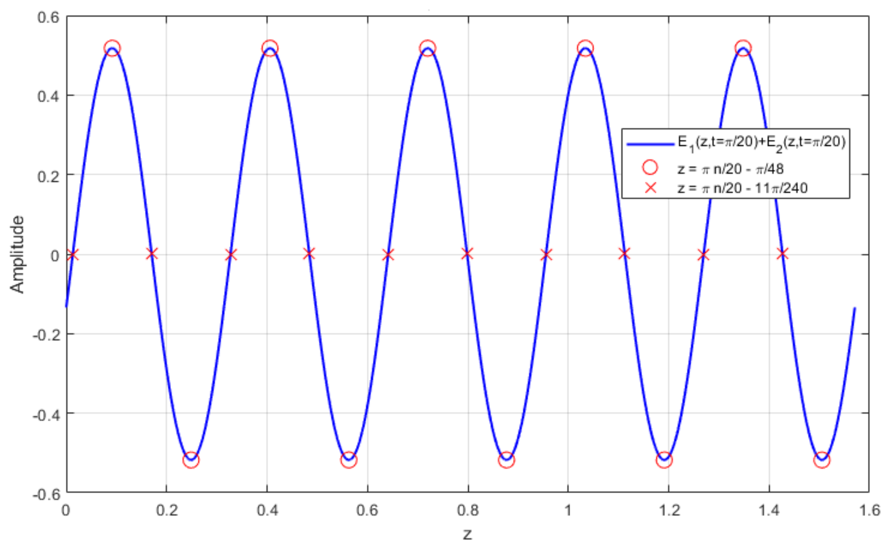
$$z = \frac{\pi n}{20} - \frac{\pi}{48}$$

Our solution for destructive interference will have the form:

$$-20z + \pi = \pi(2n+1) + 20z + \frac{11\pi}{6}$$

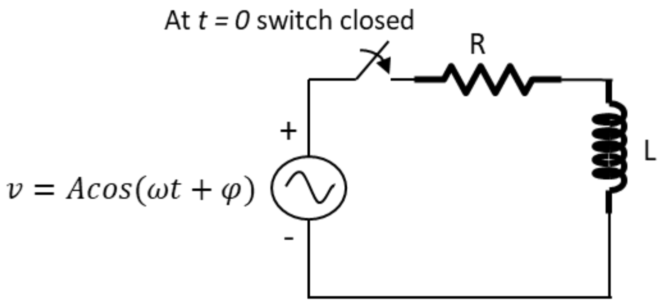
$$z = \frac{\pi n}{20} - \frac{11\pi}{240}$$

We can check our solution by graphing the superposition of the two waves:



Problem 2: Linear System - Time and Phasor Domain [30 pts]

Consider the simple circuit shown in the diagram below with a resistor R and an inductor L in series with a voltage source: $v(t) = Acos(\omega t + \phi)$ and a switch. Initially the switch is open. At $t = 0$, the switch is closed.



(a) **Write the differential equation governing the behavior of the current $i(t)$ through the circuit for $t > 0$.** Using KVL:

$$v_R(t) = Ri(t)$$

$$v_L(t) = L\frac{\partial i(t)}{\partial t}$$

$$v_i(t) = Ri(t) + L\frac{\partial i(t)}{\partial t}$$

(b) **Develop a solution to this differential equation in the time domain. Make an attempt toward obtaining an expression for i(t) for all time. You may not be able to obtain the complete solution - please show all your work in the process.**

$$v(t) = \begin{cases} \frac{\partial i}{\partial t} + \frac{R}{L}i = 0 & t \leq 0 \\ \frac{\partial i}{\partial t} + \frac{R}{L}i = \frac{A}{L}cos(\omega t + \phi) & t > 0 \end{cases}$$

Solve for $i(t)$ when $t > 0$,:

$$\frac{\partial i}{\partial t} + \frac{R}{L}i = \frac{A}{L}cos(\omega t + \phi)$$

$$e^{\frac{R}{L}t}\frac{\partial i}{\partial t} + e^{\frac{R}{L}t}\frac{R}{L}i = e^{\frac{R}{L}t}\frac{A}{L}cos(\omega t + \phi)$$

Define $\tau = \frac{L}{R}$:

$$e^{\frac{t}{\tau}}\frac{\partial i}{\partial t} + e^{\frac{t}{\tau}}\frac{1}{\tau}i = e^{\frac{t}{\tau}}\frac{A}{L}cos(\omega t + \phi)$$

$$\frac{\partial}{\partial t}\left(i(t)e^{\frac{t}{\tau}}\right) = e^{\frac{t}{\tau}}\frac{A}{L}cos(\omega t+\phi)$$
$$\int_0^t \frac{\partial}{\partial t}\left(i(t)e^{\frac{t}{\tau}}\right)dt = \int_0^t e^{\frac{t}{\tau}}\frac{A}{L}cos(\omega t+\phi)dt$$
$$i(t)e^{\frac{t}{\tau}} - i(0) = \int_0^t e^{\frac{t}{\tau}}\frac{A}{L}cos(\omega t+\phi)dt$$

Substitute i(0) = 0:

$$i(t) = e^{\frac{-t}{\tau}}\frac{A}{L}\int_0^t e^{\frac{t}{\tau}}cos(\omega t+\phi)dt \tag{2}$$

Solving to this point is sufficient for full credit.

Second order integration by parts:

$$\int_0^t e^{\frac{t'}{\tau}}cos(\omega t'+\phi)dt' = \tau e^{\frac{t'}{\tau}}cos(\omega t'+\phi)|_{t'=0}^{t'=t} - \tau\int_0^t e^{\frac{t'}{\tau}}(-\omega)sin(\omega t'+\phi)dt'$$
$$= \left(\tau e^{\frac{t}{\tau}}cos(\omega t+\phi) - \tau cos(\phi)\right) + \omega\tau\left(\tau e^{\frac{t'}{\tau}}sin(\omega t'+\phi)|_{t'=0}^{t'=t} - \omega\tau\int_0^t e^{\frac{t'}{\tau}}cos(\omega t'+\phi)dt'\right)$$
$$= \left(\tau e^{\frac{t}{\tau}}cos(\omega t+\phi) - \tau cos(\phi)\right) + \omega\tau\left(\tau e^{\frac{t}{\tau}}sin(\omega t+\phi) - \tau sin(\phi)\right) - \omega^2\tau^2\int_0^t e^{\frac{t'}{\tau}}cos(\omega t'+\phi)dt'$$

Move the last part to the left side of the equation:

$$(1+\omega^2\tau^2)\int_0^t e^{\frac{t'}{\tau}}cos(\omega t'+\phi)dt' = \tau e^{\frac{t}{\tau}}(cos(\omega t+\phi)+\omega\tau sin(\omega t+\phi)) - \tau(cos(\phi)+\omega\tau sin(\phi))$$

Define:

$$tan(\Theta) = \omega\tau$$
$$sin(\Theta) = \frac{\omega\tau}{\sqrt{1+\omega^2\tau^2}}$$
$$cos(\Theta) = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$
$$\int_0^t e^{\frac{t'}{\tau}}cos(\omega t'+\phi)dt' = \frac{\tau}{1+\omega^2\tau^2}\sqrt{1+\omega^2\tau^2}\left(e^{\frac{t}{\tau}}cos(\omega t+\phi-\Theta) - cos(\phi-\Theta)\right)$$

Plug back into Eq.2:

$$i(t) = \frac{\frac{A}{L}\tau}{\sqrt{1+\omega^2\tau^2}}\left(cos(\omega t+\phi-\Theta) - e^{\frac{-t}{\tau}}cos(\phi-\Theta)\right)$$

(c) **For the same circuit and conditions, obtain the phasor expression for the current.**

$$v(t) = Re\{\tilde{V}e^{j\omega t}\} \Rightarrow \tilde{V} = Ae^{j\phi}$$
$$i(t) = Re\{\tilde{I}e^{j\omega t}\}$$
$$Z_R = R, Z_L = j\omega L$$

Using KVL:

$$\tilde{V} = R\tilde{I} + j\omega L\tilde{I} = (R+j\omega L)\tilde{I}$$

(d) **Solve this equation for the phasor current**

$$\tilde{I} = \frac{\tilde{V}}{R + j\omega L} = \frac{A}{R + j\omega L} e^{j\phi}$$

(e) **Transform back to the time function for the current $i(t)$.**

$$i(t) = Re\left\{\frac{A}{R + j\omega L} e^{j\phi} e^{j\omega t}\right\}, \Theta = tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$\tilde{I} = \frac{A}{R\sqrt{1 + \omega^2 \frac{L^2}{R^2}} e^{j\Theta}} e^{j\phi}$$

$$i(t) = Re\left\{\frac{A}{R\sqrt{1 + \omega^2 \frac{L^2}{R^2}} e^{j\Theta}} e^{j\phi} e^{j\omega t}\right\}$$

$$i(t) = \frac{A}{R\sqrt{1 + \omega^2 \frac{L^2}{R^2}}} cos(\omega t + \phi - \Theta)$$

Recall that $\tau = \frac{L}{R}$:

$$i(t) = \frac{\frac{A}{L}\tau}{\sqrt{1 + \omega^2 \tau^2}} cos(\omega t + \phi - \Theta)$$

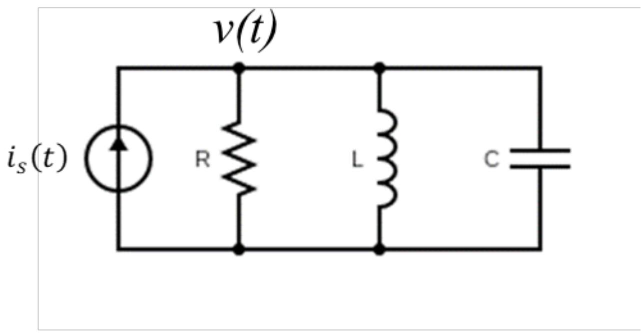
(f) **Comment on the methodology of the time domain versus using the phasor domain to obtain the solution to this linear system.**

The first term of the expression found in part b is the same as the solution found in part e. This term is the steady state solution, the solution as $t \rightarrow \tau$. The second term in the expression for part b is the transient solution, which has no t dependence. When we only care about the steady state solution, phasor domain solutions are far easier to work through.

Problem 3: Circuit Analysis using the Phasor Domain [40 pts]

Consider a parallel RLC circuit shown in the figure below and driven by the current source:

$$i_s(t) = I_0 cos(\omega t + \frac{\pi}{3})[\mathrm{A}]$$



(a) **Obtain the equation for $v(t)$ in terms of R, L, C, and $i_s(t)$.**

Using KCL:

$$i_s(t) = i_C(t) + i_R(t) + i_L(t)$$

$$i_s(t) = C\frac{\partial v(t)}{\partial t} + \frac{v(t)}{R} + \frac{1}{L}\int_0^t v(t')dt'$$

(b) **Convert this differential equation to the phasor domain.**

$$\tilde{I}_s = j\omega C\tilde{V} + \frac{1}{R}\tilde{V} + \frac{1}{j\omega L}\tilde{V} = \tilde{V}\left(j\omega C + \frac{1}{R} + \frac{1}{j\omega L}\right)$$

where $\tilde{I}_s = I_0 e^{j\frac{\pi}{3}}$

(c) **Solve the equation for $\tilde{V}$, the steady-state phasor voltage across each component.**

$$\tilde{V} = \frac{\tilde{I}_s}{j\omega C + \frac{1}{R} + \frac{1}{j\omega L}} = \frac{j\omega RL}{R - \omega^2 RLC + j\omega L}\tilde{I}_s$$

(d) **Using the phasor domain analysis obtain the full time domain expression $v(t)$.**

$$\tilde{V} = \frac{j\omega RL}{R - \omega^2 RLC + j\omega L}\tilde{I}_s$$

$$\tilde{V} = \frac{\omega RL e^{j\frac{\pi}{2}}}{\sqrt{(R - \omega^2 RLC)^2 + (\omega L)^2 e^{j\theta}}}\tilde{I}_s$$

where

$$\theta = tan^{-1}\left(\frac{\omega L}{R - \omega^2 RLC}\right)$$

$$\tilde{V} = \frac{\omega RL}{\sqrt{(R - \omega^2 RLC)^2 + (\omega L)^2}} e^{j(\frac{\pi}{2} - \theta)} \tilde{I}_s$$

$$v(t) = \frac{\omega RLI_0}{\sqrt{(R - \omega^2 RLC)^2 + (\omega L)^2}} cos(\omega t + \frac{\pi}{3} - \Theta)$$

where $\Theta = \frac{\pi}{2} - \theta$